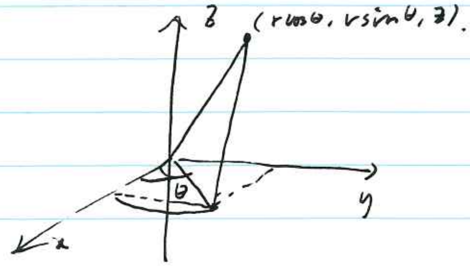
LEC 12 Laplace equation in a circular cylinder.

cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$



Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

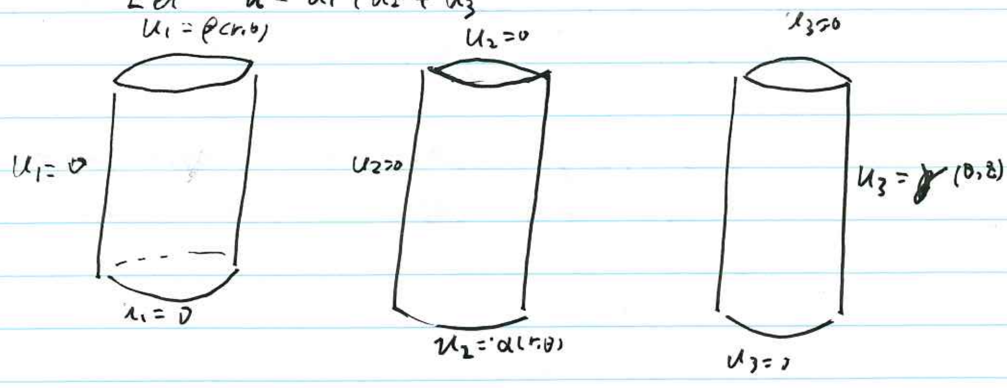
BCs:

Top: $u(r,\theta,H) = \beta(r,\theta)$

bottom $u(r,\theta,0) = \alpha(r,\theta)$.

lateral side: $u(a,\theta,z) = \gamma(\theta,z)$.

Let $u = u_1 + u_2 + u_3$



Separation of variables.

$$u(r,\theta,z) = f(r)\,g(\theta)\,h(z).$$

$$\frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{r^2}\frac{1}{g}\frac{d^2g}{d\theta^2} + \frac{1}{h}\frac{d^2h}{dz^2} = 0$$

~~Bou~~ z - problem. $\quad \frac{1}{h}\frac{d^2h}{dz^2} = \lambda$

Do we expect oscillations in z?

\

r and θ parts?

Multiply by r^2

$$\frac{r^2}{f}\cdot\frac{d}{dr}\left(r\frac{df}{dr}\right)+\lambda r^2=-\frac{1}{g}\frac{d^2g}{d\theta^2}=\mu$$

We expect eigenvalue problems in θ, so $\mu \geq 0$. Let $m^2=\mu$,
$m=0,1,2,\cdots$.

Now $g''+\mu g=0 \Rightarrow \sin m\theta \quad \cos m\theta$.

The other two problems (eigenvalue problems).

$$\frac{d^2h}{dz^2}=\lambda h$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right)+(\lambda r^2-m^2)f=0.$$

Only one of the equation will become the eigenvalue problem for λ. As usual, we need $|f(0)|<\infty$.

1. zero temperature on the lateral sides and on the bottom or top. (u_1).

$$\begin{cases} \Delta u_1=0. \\ u_1(r,\theta,0)=0. \\ u_1(r,\theta,H)=\beta(r,\theta) \\ u_1(a,\theta,z)=0. \end{cases}$$

r-problem $\quad f(a)=0, \quad |f(0)|<\infty \quad$ (two homogeneous B.C.s).

By Rayleigh quotient, $\lambda>0$.

$f=C_1 J_m(\sqrt{\lambda}r)+C_2 Y_m(\sqrt{\lambda}r)$.

$C_2=0$ due to $|f(0)|<\infty$.

Hence $J_m(\sqrt{\lambda}a)=0$ by another BC.

Now since $\lambda>0$, $h(0)=0 \Rightarrow h=\sinh(\sqrt{\lambda}z)$

product solution.

$$\sinh\sqrt{\lambda_{mn}}z\, J_m(\sqrt{\lambda_{mn}}r)\begin{cases}\sin m\theta\\ \cos m\theta\end{cases}$$

General solution.

$$u(r,\theta,z)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sinh\sqrt{\lambda_{mn}}z\, J_m(\sqrt{\lambda_{mn}}r)\cos m\theta$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sinh\sqrt{\lambda_{mn}}z\, J_m(\sqrt{\lambda_{mn}}r)\sin m\theta$$

2. Zero temperature on the top and bottom

$$\begin{cases}\Delta u_3=0\\ u_3(r,\theta,0)=0=u_3(r,\theta,H)\\ u(a,\theta,z)=\gamma(\theta,z)\end{cases}$$

z-direction has two homogeneous BCs.

$$\begin{cases}h''=\lambda h\\ h(0)=h(H)=0\end{cases}\Rightarrow\quad \lambda=-\left(\frac{n\pi}{H}\right)^2\quad n=1,2,\cdots\quad h(z)=\sin\frac{n\pi z}{H}$$

We have both oscillations in z and θ.
For r.

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(-\left(\frac{n\pi}{H}\right)r^2-m^2\right)f=0.$$

$\Rightarrow$ not Bessel.

Let $s=i\left(\frac{n\pi}{H}\right)r$

$$s^2 f'' + s f' + (s^2 - m^2) f = 0.$$

$$f = c_1 J_m(s) + c_2 Y_m(s)$$

not quite useful.

Instead, let $w = \frac{n\pi}{H} r$.

$$w^2 \frac{d^2 f}{dw^2} + w \frac{df}{dw} + (-w^2 - m^2) f = 0$$: modified bessel fn.

$$f = C_1 K_m\left(\frac{n\pi}{H} r\right) + C_2 I_m\left(\frac{n\pi}{H} r\right).$$

K_m is singular near $r=0$.

$\therefore$ product sol. is

$$I_m\left(\frac{n\pi}{H} r\right) \sin \frac{n\pi z}{H} \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix}$$

$$\therefore u_3(r, \theta, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} E_{mn} I_m\left(\frac{n\pi}{H} r\right) \sin \frac{n\pi z}{H} \cos m\theta$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} I_m\left(\frac{n\pi r}{H}\right) \sin \frac{n\pi z}{H} \sin m\theta$$

Spherical problems.

3D wave equation describes the vibrations of earth.
Geophysics $\Rightarrow$ earthquake.

Compressional waves : P primary (small)
S shear waves (later)
L long period surface waves

BC. $u(r, \theta, \phi, 0) = F(\rho, \theta, \phi)$

$\frac{\partial u}{\partial t}(r, \theta, \phi, 0) = G(\rho, \theta, \phi).$

$x = \rho \sin\varphi \cos\theta$
$y = \rho \sin\varphi \sin\theta$
$z = \rho \cos\varphi$

Seperation of variables.

$u(\rho, \theta, \varphi, t) = w(\rho, \theta, \varphi) h(t)$.

$$\begin{cases} \dfrac{d^2 h}{dt^2} = -\lambda c^2 h. \\ \Delta w + \lambda w = 0. \end{cases}$$

Laplacian in spherical coord.

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial w}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\varphi} \frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial w}{\partial \varphi}\right) + \frac{1}{\rho^2 \sin^2\varphi} \frac{\partial w}{\partial \theta^2} + \lambda w = 0.$$

another separation.

$$w(\rho, \theta, \varphi) = f(\rho) q(\theta) g(\varphi)$$

Note the coefficients does not depend on θ

The eigenfunctions in θ are $\cos m\theta$, $\sin m\theta$.

then $\frac{\partial^2 w}{\partial \theta^2} = -m^2 w$.

Multiply by ρ^2

$$\frac{1}{f} \frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \lambda \rho^2 = -\frac{1}{g \sin\varphi} \frac{d}{d\varphi}\left(\sin\varphi \frac{dg}{d\varphi}\right) + \frac{m^2}{\sin^2\varphi}$$

$= \mu$.

Two problems.

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + (\lambda\rho^2 - \mu)f = 0 \qquad ①$$

$$\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \left(\mu\sin\phi - \frac{m^2}{\sin\phi}\right)g = 0. \qquad ②$$

Both of them are not regular S-L.

② Let $x = \cos\phi$.

$$\frac{d}{d\phi} = \frac{dx}{d\phi}\, d\phi = -\sin\phi \frac{d}{dx}.$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dg}{dx}\right] + \left(\mu - \frac{m^2}{1-x^2}\right)g = 0.$$

$\Rightarrow \mu = n(n+1)$ $\Rightarrow$ associated Legendre functions
(spherical harmonics)
$P_n^m(x)$ $Q_n^m(x)$.
$P_n^m(x)$ is bounded at $x = \pm 1$

$m=0$: Legendre polynomials.

$$\frac{d}{dx}\left[(1-x^2)\frac{dg}{dx}\right] + n(n+1)g = 0.$$

$\Rightarrow$ Legendre polynomials.

$n=0$: $P_0(x) = 1$

$n=1$ $P_1(x) = x = \cos\phi$

$n=2$ $P_2(x) = \frac{1}{2}(3x^2-1) = \frac{1}{4}(3\cos 2\phi + 1).$

Rodrigues formula.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

<u>Radial problem.</u>

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \left(\lambda \rho^2 - n(n+1)\right) f = 0 .$$

$\Rightarrow$ spherical Bessel functions.

$$f(\rho) = \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho) .$$

BC $\Rightarrow$. $J_{n+\frac{1}{2}}(\sqrt{\lambda} a) = 0$

Product solutions.

$$u(\rho, \theta, \phi, t) = \left\{ \begin{matrix} \cos c\sqrt{\lambda} t \\ \sin c\sqrt{\lambda} t \end{matrix} \right\} \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho) \left\{ \begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix} \right\} P_n^m(\cos\phi) .$$